OPTIMAL STEP-STRESS TESTING UNDER PROGRESSIVE TYPE-I CENSORING

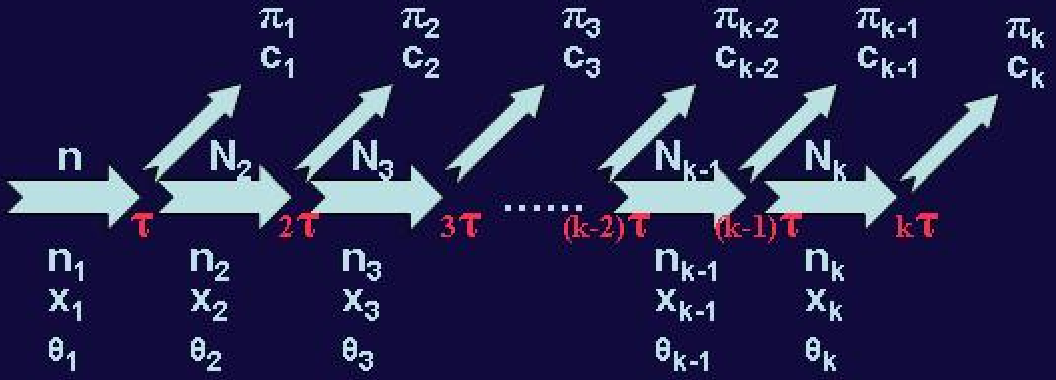
David Han
SOSGSSD 2006
McMaster University

OUTLINE

1 INTRODUCTION

Why ALT (Accelerated Life Testing) ?

- highly reliable products with long life-spans $\Longrightarrow$ time-consuming and costly tests (*e.g.*, in developing prototypes)
- the units are subjected to higher stress levels for rapid failures in ALT
- quicker collection of information on the life distribution
- a special class of ALT is the STEP-STRESS TEST: gradual increase of the stress at some time points during the test



Why PC (Progressive Censoring) ?

- the reasons of cost reduction and time constraint
- the efficient exploitation of the available resources
- withdrawn units are used in other experiments

PAST WORKS ON OPTIMAL ALT

- the cumulative exposure model was proposed by Nelson (1980).
- Miller and Nelson (1983) initiated research by assuming the exponential distribution and complete failure data under a simple step-stress model.
- Bai, Kim, & Lee (1989) extended the results to the case of time-censored data.
- the case of three stress levels was dealt by Khamis & Higgins (1996).

- **Khamis and Higgins (1997) also considered the problem under a Weibull distribution.**

- **Khamis (1998) undertook some numerical investigation for the general k-level, M-variable case.**

- **inference issues with the cumulative exposure model under exponentiality were studied by Xiong (1998), Xiong & Milliken (1999).**

- **Gouno, Sen, & Balakrishnan (2004) tackled the selection problem of optimal stress change points for a general k-level case with the large sample assumption and progressively Type-I censored data.**

The main objective is . . .

- to readdress the optimality problem under the large sample assumption,
- to investigate the choice of an optimal time point for stress change when the sample size is small to moderate,
- to suggest some practical modifications for a feasible step-stress analysis under a PC scheme in the case of a small sample.

We consider the equispaced step with a single τ, the duration of each testing stage.

2 MODEL DESCRIPTIONS AND OPTIMALITY CRITERIA

Assumptions **:**

(i) A cumulative exposure model holds.

(ii) For any stress level,
the lifetime of a unit $\sim$ Exponential.

(iii) At stress level x_i, the MTTF of a unit is a log-linear function of stress:

$$\log \theta_i = \alpha + \beta x_i,$$

where the regression parameters α and β are unknown.

2.1 k-LEVEL STEP-STRESS WITH A LARGE SAMPLE UNDER PROGRESSIVE CENSORING

We need to assume large n, small π_i's, and small k to ensure a sufficient number of surviving items to be censored at the end of each step.

$$f(t) \;=\; f_i(t-(i-1)\tau)\prod_{j=1}^{i-1} S_j(\tau),$$

$$\textbf{if}\;\begin{cases}(i-1)\tau \le t \le i\tau & \textbf{for } i = 1, 2, \ldots, k-1\\ (k-1)\tau \le t < \infty & \textbf{for } i = k\end{cases} \qquad (1),$$

$$\textbf{where } f_i(t) = \frac{1}{\theta_i}\exp\left(-\frac{t}{\theta_i}\right) \textbf{ for } i = 1, 2, \ldots, k\textbf{.}$$

$$
\begin{aligned}
F(t) \; &= \; 1 - \left[\prod_{j=1}^{i-1} S_j(\tau) \right] S_i(t-(i-1)\tau), \\
& \qquad\qquad \textbf{if} \; \begin{cases} (i-1)\tau \le t \le i\tau & \textbf{for } i = 1, 2, \ldots, k-1 \\ (k-1)\tau \le t < \infty & \textbf{for } i = k \end{cases}
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
F_i(t) \; &= \; 1 - \exp\left(-\frac{t}{\theta_i} \right), \\
S_i(t) \; &= \; 1 - F_i(t) = \exp\left(-\frac{t}{\theta_i} \right).
\end{aligned}
$$

Lemma 2.1.1. *The JPDF of* $\mathbf{y}$ *and* $\mathbf{n}$ *is*

$$f_J(\mathbf{y}, \mathbf{n}) = \left[\prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!}\right]\left[\prod_{i=1}^{k} \theta_i^{-n_i}\right] \exp\left(-\sum_{i=1}^{k} \frac{U_i}{\theta_i}\right),$$

where

$$U_i = \sum_{j=1}^{n_i} (y_{i,j} - (i-1)\tau) + (N_i - n_i)\tau, \qquad i = 1, 2, \ldots, k.$$

Note that U_i **is precisely the** *Total Time on Test* **statistic at stress level** x_i**.**

Lemma 2.1.2. *The MLEs $\hat{\alpha}$ and $\hat{\beta}$ are obtained as simultaneous solutions to the following two non-linear equations:*

$$\hat{\alpha} = \log\left(\frac{\sum_{i=1}^{k} U_i \exp\left(-\hat{\beta}x_i\right)}{\sum_{i=1}^{k} n_i}\right),$$

$$\left[\sum_{i=1}^{k} n_i\right]\left[\sum_{i=1}^{k} U_i x_i \exp\left(-\hat{\beta}x_i\right)\right] - \sum_{i=1}^{k} n_i x_i \sum_{i=1}^{k} U_i \exp\left(-\hat{\beta}x_i\right) = 0.$$

Non-linearity of $\hat{\alpha}$ and $\hat{\beta}$ $\Longrightarrow$ virtually impossible to find their exact marginal/joint distributions for exact inference

$\therefore$ statistical inference on the MLEs are based on the asymptotic distributional result.

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \sim BVN\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix},\ [\mathbf{I}_n(\alpha, \beta)]^{-1} \right) \qquad \textbf{as } n \to \infty$$

Theorem 2.1.1. *Given* $n_1, n_2, \ldots, n_{i-1}$, *the random variable* n_i *has a binomial distribution with parameters* $(N_i, F_i(\tau))$ *for* $i = 1, 2, \ldots, k$.

Corollary 2.1.1. *For* $i = 1, 2, \ldots, k$,

$$E[n_i] = E[N_i]F_i(\tau).$$

Theorem 2.1.2. *Given* $n_1, n_2, \ldots, n_i$, *the random variables* $(y_{i,j} - (i-1)\tau)$, $j = 1, 2, \ldots, n_i$, *constitute the order statistics from a random sample of size* n_i *with a right-truncated exponential distribution whose PDF is defined as*

$$f_{i,\tau}(z) = \begin{cases} \dfrac{f_i(z)}{F_i(\tau)}, & 0 \le z \le \tau \\ 0, & otherwise \end{cases} = \begin{cases} \dfrac{e^{-z/\theta_i}}{\theta_i(1 - e^{-\tau/\theta_i})}, & 0 \le z \le \tau \\ 0, & otherwise \end{cases}, \tag{3}$$

for $i = 1, 2, \ldots, k$.

Corollary 2.1.2. *For* $i = 1, 2, \ldots, k$,

$$E\left[\sum_{l=1}^{n_i}(y_{i,j} - (i-1)\tau)\right] = E[N_i](\theta_i F_i(\tau) - \tau S_i(\tau)).$$

Lemma 2.1.3. *For* $i = 1, 2, \ldots, k$,

$$E[N_i] = n\left[1 - \sum_{j=1}^{i-1}\frac{\pi_j}{G_j(\tau)}\right]G_{i-1}(\tau),$$

where

$$G_j(\tau) = G_{j-1}(\tau)S_j(\tau) = \prod_{i=1}^{j} S_i(\tau).$$

Theorem 2.1.3. *The expected information matrix of α and β is*

$$\mathbf{I}_n(\alpha,\beta) = n\begin{pmatrix} \sum_{i=1}^{k} A_i(\tau) & \sum_{i=1}^{k} A_i(\tau)x_i \\ \sum_{i=1}^{k} A_i(\tau)x_i & \sum_{i=1}^{k} A_i(\tau)x_i^2 \end{pmatrix},$$

where

$$A_i(\tau) = \left[1 - \sum_{j=1}^{i-1} \frac{\pi_j}{G_j(\tau)}\right] G_{i-1}(\tau) F_i(\tau).$$

2.2 OPTIMALITY CRITERION FUNCTIONS AND EXISTENCE OF OPTIMAL STRESS CHANGE POINT

- $A_i(\tau)$ **exemplifies the complexity introduced by the PC scheme.**
- **For certain values of** τ**,** $A_i(\tau)$ **can be negative giving rise to disconcerting anomalies such as a negative determinant of** $\mathbf{I}_n(\alpha, \beta)$ **or a negative variance function.**

$\therefore$ **We need to confine the search for optimal** τ **to the region**

$$\mathcal{C}_\tau = \{\tau : A_i(\tau) > 0, \quad i = 2, 3, \ldots, k\}.$$

2.2.1 V-OPTIMALITY

$$
\begin{aligned}
\phi(\tau) &= n \cdot \mathrm{AVar}(\log \hat{\theta}_0) = n \cdot \mathrm{AVar}(\hat{\alpha} + \hat{\beta} x_0) \\
&= n \cdot (1, x_0) \mathbf{I}_n^{-1}(\alpha, \beta) \begin{pmatrix} 1 \\ x_0 \end{pmatrix} \\
&= \frac{2 \cdot \displaystyle\sum_{i=1}^{k} A_i(\tau)(x_i - x_0)^2}{\displaystyle\sum_{i=1}^{k} \sum_{j=1}^{k} A_i(\tau) A_j(\tau)(x_i - x_j)^2} \qquad (4)
\end{aligned}
$$

The V-optimal τ ($viz.$, τ_V^*) minimizes $\phi(\tau)$ to estimate the MTTF of a unit at the use-condition ($viz.$, θ_0) with maximum precision and minimum variability.

2.2.2 D-OPTIMALITY

- **the overall volume of the asymptotic joint confidence region of** $(\alpha, \beta) \propto |\mathbf{I}_n^{-1}(\alpha, \beta)|^{1/2}$
- **a larger value of** $|\mathbf{I}_n(\alpha, \beta)| \Longrightarrow$ **a smaller asymptotic joint confidence ellipsoid of** $(\alpha, \beta) \Longrightarrow$ **a higher joint precision of** $(\hat{\alpha}, \hat{\beta})$

The D-optimal τ **(***viz.***,** τ_D^***) maximizes**

$$\begin{aligned} \delta(\tau) &= n^{-2}|\mathbf{I}_n(\alpha, \beta)| \\ &= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} A_i(\tau)A_j(\tau)(x_i - x_j)^2. \end{aligned} \tag{5}$$

2.2.3 A-OPTIMALITY

- **the sum of marginal Fisher information terms of the parameters ≡ the sum of the diagonal elements or trace of** $\mathbf{I}_n(\alpha, \beta)$
- **a general measure of the size of** $\mathbf{I}_n(\alpha, \beta)$

$$\begin{aligned} a(\tau) &= \frac{1}{n} tr(\mathbf{I}_n(\alpha, \beta)) \\ &= \sum_{i=1}^{k} A_i(\tau) + \sum_{i=1}^{k} A_i(\tau) x_i^2 = \sum_{i=1}^{k} A_i(\tau)(1 + x_i^2). \end{aligned} \tag{6}$$

The A-optimal τ **(***viz.*, τ_A^***) maximizes** $a(\tau)$**.**

Proposition 2.2.1. *In the case of the simple step-stress testing with progressive Type-I censoring, there exists an optimal step duration* τ_V^* *which is the solution of* $\phi'(\tau) = 0$.

Proposition 2.2.2. *In the case of the simple step-stress test under progressive Type-I censoring, the D-optimal stress change point* τ_D^* *is the solution of* $A_1'(\tau)A_2(\tau) + A_1(\tau)A_2'(\tau) = 0$.

Proposition 2.2.3. *For the simple step-stress test with progressive Type-I censoring, the A-optimal stress change point* τ_A^* *exists when* $\dfrac{x_2^2 - x_1^2}{1 + x_2^2} > \pi_1^{\theta_1/\theta_2}$, *and it is the solution of* $a'(\tau) = 0$. *Otherwise,* $\tau_A^* = -\theta_1 \log \pi_1$.

2.3 k-LEVEL STEP-STRESS WITH A SMALL SAMPLE UNDER PROGRESSIVE CENSORING

PROBLEMS OF THE ASYMPTOTIC MODEL:

- a small sample size in a real reliability experiment
- severe censoring due to various reasons including the budgetary constraints and facility requirements

$\therefore$ We need a modification to guarantee a feasible PC scheme.

One such modification is to decide on a fixed proportion of unfailed items to be removed at the end of each stage. First, define a vector of fixed proportions

$$\pi^* = (\pi_1^*, \pi_2^*, \ldots, \pi_{k-1}^*),$$

where $0 \leq \pi_i^* < 1$ for $i = 1, 2, \ldots, k-1$. The number of censored items at the end of stress level x_i is

$$c_i = \Upsilon((N_i - n_i)\pi_i^*) \qquad \textbf{for } i = 1, 2, \ldots, k-1,$$

where $\Upsilon(\cdot)$ is a discretizing function of one's choice, mapping its argument to a non-negative integer ($e.g.$, $round(\cdot)$, $floor(\cdot)$, $ceiling(\cdot)$, $trunc(\cdot)$, etc).

Under this modification, we allow the life test to terminate before reaching the last stress level x_k.

- **we shall assume**

$$c_i = (N_i - n_i)\pi_i^* \qquad \textbf{for } i = 1, 2, \ldots, k-1.$$

- **when** $\pi^* = (0, 0, \ldots, 0) = \mathbf{0_{k-1}}$**, we have** $\mathbf{c} = \mathbf{0_{k-1}}$ **and** $\pi = \mathbf{0_{k-1}}$**, and it corresponds to the case of a** k**-level step-stress testing under Type-I right censoring with** $c_k = n - \sum_{j=1}^{k} n_j$**.**
- **if** $c_k > 0$ **or** $n_k > 0$ **(equivalently,** $N_k = n_k + c_k > 0$**), it implies that the life test has proceeded to the last stress level** x_k**.**

Lemma 2.3.1. *For* $i = 1, 2, \ldots, k$,

$$E[N_i] = n \prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*).$$

Theorem 2.3.1. *Under the proposed modification, the expected information matrix of α and β is*

$$\mathbf{I}_n(\alpha,\beta) = n \begin{pmatrix} \sum_{i=1}^{k} A_i(\tau) & \sum_{i=1}^{k} A_i(\tau)x_i \\ \sum_{i=1}^{k} A_i(\tau)x_i & \sum_{i=1}^{k} A_i(\tau)x_i^2 \end{pmatrix},$$

where

$$A_i(\tau) = F_i(\tau)\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*).$$

2.4 OPTIMALITY CRITERION FUNCTIONS AND EXISTENCE OF OPTIMAL STRESS CHANGE POINT

- $A_i(\tau) > 0$ **for all** $\tau > 0 \Longrightarrow$ **eliminates any disconcerting anomalies**
- **since the censoring is based on the number of surviving units at the end of each stage, censoring beyond what is available on test is prevented.**

$\therefore$ **no restriction on the search region for optimal** τ **after modification** $(i.e., \mathcal{C}_\tau = \{\tau : \tau > 0\})$**.**

Proposition 2.4.1. *In the case of the simple step-stress testing with progressive Type-I censoring, there exists an optimal step duration* τ_V^* *which is the solution of* $\phi'(\tau) = 0$.

Proposition 2.4.2. *In the case of the simple step-stress test under progressive Type-I censoring, the D-optimal stress change point* τ_D^* *is the solution of* $A_1'(\tau)A_2(\tau) + A_1(\tau)A_2'(\tau) = 0$.

Proposition 2.4.3. *For the simple step-stress test with progressive Type-I censoring, the A-optimal stress change point is*

$$\tau_A^* = \theta_2 \log\left[\left(1+\frac{\theta_1}{\theta_2}\right)(1-Q_1^A)^{-1}\right] \text{ where } Q_1^A = \frac{1+x_1^2}{(1-\pi_1^*)(1+x_2^2)},$$

and it exists when $\dfrac{x_2^2 - x_1^2}{1+x_2^2} > \pi_1^*$. *Otherwise,* τ_A^* *does not exist.*

2.5 OTHER MODIFICATION FOR A VIABLE k-LEVEL STEP-STRESS TESTING UNDER PC

One may want to censor a pre-determined number of units instead of a proportion of live units at the end of each stage so that the experimenter knows how many units would be freed by censoring at the end of the current stage given that the test should proceed to the next stress level. We first define

$$\mathbf{c}^* = (c_1^*, c_2^*, \ldots, c_{k-1}^*),$$

where c_i^* is the fixed number of units to be removed at the end of stress level x_i for $i = 1, 2, \ldots, k-1$.

The actual number of progressively censored units at stress level x_i is

$$\begin{aligned} c_i &= \min\{c_i^*,\ N_i - n_i\} \\ &= \min\left\{c_i^*,\ n - \sum_{j=1}^{i} n_j - \sum_{j=1}^{i-1} c_j\right\}, \end{aligned} \tag{7}$$

for $i = 1, 2, \ldots, k-1$. We take $c_k = N_k - n_k$ as before.

- **when $\mathbf{c}^* = (0, 0, \ldots, 0) = \mathbf{0_{k-1}}$, we have $\mathbf{c} = \mathbf{0_{k-1}}$ and $\pi = \mathbf{0_{k-1}}$ as well. Then, it is the case of a k-level step-stress testing under Type-I right censoring with $c_k = n - \sum_{j=1}^{k} n_j$.**
- **when $c_k > 0$ or $n_k > 0$ (equivalently, $N_k = n_k + c_k > 0$), the life test has proceeded to the last stress level x_k.**

Lemma 2.5.1. *$E[N_1] = n$, and for $i = 1, 2, \ldots, k-1$,*

$$E[N_{i+1}] = \sum_{n_1=0}^{\eta_{i,1}} \sum_{n_2=0}^{\eta_{i,2}} \cdots \sum_{n_{i-1}=0}^{\eta_{i,i-1}} \Big[(N_i^* - c_i^*) B_{N_i^*}^{[i]}(\eta_{i,i}) \\ - N_i^* F_i(\tau) B_{N_i^*-1}^{[i]}(\eta_{i,i} - 1) \Big] p_J(n_1, n_2, \ldots, n_{i-1}),$$

where

$$\begin{aligned}
\eta_{i,l} &= \eta_{i,l-1} - n_{l-1} = n - \sum_{j=1}^{l-1} n_j - \sum_{j=1}^{i} c_j^* - 1, \\
&\hspace{12em} \textit{for } l = 1, 2, \ldots, i, \\
N_i^* &= N_{i-1}^* - n_{i-1} - c_{i-1}^* = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} c_j^*, \\
B_N^{[i]}(x) &= Pr(X \leq x) \qquad \textit{wherein } X \sim Binomial(N, F_i(\tau)) \\
&= \sum_{j=0}^{x} \binom{N}{j} [F_i(\tau)]^j [S_i(\tau)]^{N-j}, \qquad 0 \leq x \leq N,
\end{aligned}$$

and $p_J(n_1, n_2, \ldots, n_{i-1})$ *is the JPMF of* $(n_1, n_2, \ldots, n_{i-1})$ *as given in corollary 2.1.2.*

Theorem 2.5.1. *The expected information matrix of α and β under the proposed modification is*

$$\mathbf{I}_n(\alpha, \beta) = n \begin{pmatrix} \sum_{i=1}^{k} A_i(\tau) & \sum_{i=1}^{k} A_i(\tau) x_i \\ \sum_{i=1}^{k} A_i(\tau) x_i & \sum_{i=1}^{k} A_i(\tau) x_i^2 \end{pmatrix},$$

where

$$A_i(\tau) = \frac{1}{n} E[n_i] = \frac{1}{n} E[N_i] F_i(\tau)$$

with $E[N_i]$ as obtained in lemma 2.5.1.

$A_i(\tau) > 0$ **for all** $\tau > 0 \Longrightarrow E[N_i] > 0$ **for** $i = 1, 2, \ldots, k$

$\therefore$ **the search region for optimal** τ **is unrestricted**

(*i.e.***,** $\mathcal{C}_\tau = \{\tau : \tau > 0\}$**).**

2.6 CONDITIONAL ANALYSIS OF k-LEVEL STEP-STRESS UNDER PROGRESSIVE CENSORING

We tackle the problem of selecting an optimal stress duration using the conditional approach. We observe that

$$\{\mathbf{n} : N_k > 0\} \subset \{\mathbf{n} : N_{k-1} > 0\} \subset \cdots \subset \{\mathbf{n} : N_1 \equiv n > 0\} = \{\mathbf{n}\}$$

and thus,

$$\begin{aligned}\{\mathbf{n} : N_2 > 0, N_3 > 0, \ldots, N_k > 0\} \\ &= \{\mathbf{n} : N_2 > 0\} \cap \{\mathbf{n} : N_3 > 0\} \cap \cdots \cap \{\mathbf{n} : N_k > 0\} \\ &= \{\mathbf{n} : N_k > 0\}.\end{aligned}$$

Condition of successful censoring at every stress level

$\equiv$ Condition of the test proceeding to the last level x_k

Lemma 2.6.1. *For* $i = 1, 2, \ldots, k-1$,

$$Pr(N_k = 0 | n_1, n_2, \ldots, n_{i-1}) = [H_i(\tau)]^{N_i},$$

where

$$H_i(\tau) = \begin{cases} F_i(\tau) + S_i(\tau)[H_{i+1}(\tau)]^{1-\pi_i^*}, & \textit{for } i = 1, 2, \ldots, k-1 \\ 0, & \textit{for } i = k \end{cases}$$

Corollary 2.6.1. *For k stress levels, the probability of a life test proceeding to stress level x_k is*

$$Pr(N_k > 0) = 1 - [H_1(\tau)]^n.$$

Theorem 2.6.1. *For* $i = 1, 2, \ldots, k$,

$$E_c[n_i] = E[n_i | N_k > 0] = E[n_i] \frac{1 - V_i(\tau)}{1 - [H_1(\tau)]^n},$$

where

$$V_i(\tau) = \begin{cases} \dfrac{[H_1(\tau)]^{n-1}}{\prod_{j=1}^{i-1} [H_{j+1}(\tau)]^{\pi_j^*}}, & \textit{for } i = 1, 2, \ldots, k-1 \\ 0, & \textit{for } i = k \end{cases}$$

and

$$E[n_i] = n \left[\prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*) \right] F_i(\tau).$$

Lemma 2.6.2. *For* $i = 1, 2, \ldots, k$*,*

$$E_c[N_i] = E[N_i | N_k > 0] = E[N_i] \frac{1 - H_i(\tau) V_i(\tau)}{1 - [H_1(\tau)]^n},$$

where $E[N_i]$ *is as given previously.*

Theorem 2.6.2. *The expected information matrix of the regression parameters, α and β, conditioned on $N_k > 0$ is*

$$\mathbf{I}_n(\alpha,\beta) = n\begin{pmatrix} \sum_{i=1}^{k} A_i(\tau) & \sum_{i=1}^{k} A_i(\tau)x_i \\ \sum_{i=1}^{k} A_i(\tau)x_i & \sum_{i=1}^{k} A_i(\tau)x_i^2 \end{pmatrix},$$

where

$$\begin{aligned} A_i(\tau) &= \frac{E[N_i]}{n(1-[H_1(\tau)]^n)}\left[(1-V_i(\tau))F_i(\tau)+\frac{\tau}{\theta_i}(1-H_i(\tau))V_i(\tau)\right] \\ &= \frac{1}{1-[H_1(\tau)]^n}\left[\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*)\right] \\ &\qquad \times\Big[(1-V_i(\tau))F_i(\tau)+\tau(1-H_i(\tau))V_i(\tau)\exp(\alpha+\beta x_i)\Big]. \end{aligned}$$

REMARK:

$0 \leq H_1(\tau) < 1$, **and so it follows immediately that**

$$\lim_{n\to\infty} Pr(N_k > 0) = 1 - \lim_{n\to\infty} [H_1(\tau)]^n = 1.$$

Then, for $i = 1, 2, \ldots, k$**,**

$$\begin{aligned} \lim_{n\to\infty} V_i(\tau) &= \frac{\lim_{n\to\infty} [H_1(\tau)]^{n-1}}{\prod_{j=1}^{i-1} [H_{j+1}(\tau)]^{\pi_j^*}} = 0, \\ \lim_{n\to\infty} E_c[n_i] &= E[n_i], \\ \lim_{n\to\infty} E_c[N_i] &= E[N_i]. \end{aligned}$$

$\therefore$ all the distributional results obtained by conditioning on $N_k > 0$ ultimately converge to the unconditional results when the sample size n gets larger.

Conditioning makes less relevance to the analysis when the initial sample size is large.

3 NUMERICAL STUDIES AND RESULTS

- **to investigate the existence of the optimal stress change points,**
- **to evaluate them as a function of varying parameters (*viz.*, the sample size, MTTF, the number of stress levels, and the degree of censoring).**

For the entire study, $x_i = x_0 + id$ with $x_0 = 10$ and $d = 5$. With this setup, optimizing with respect to either of the V-optimality or D-optimality criterion is independent of x_0 and d even though the A-optimality criterion is sensitive to the choice of x_0 and d.

We also set

$$\theta_{i+1} = \rho\theta_i, \qquad i = 1, 2, \ldots, k-1, \qquad 0 < \rho < 1,$$

with selected values of θ_1 and ρ. Using this formula, a decreasing geometric sequence of MTTF is simulated with an increasing arithmetic sequence of stress levels.

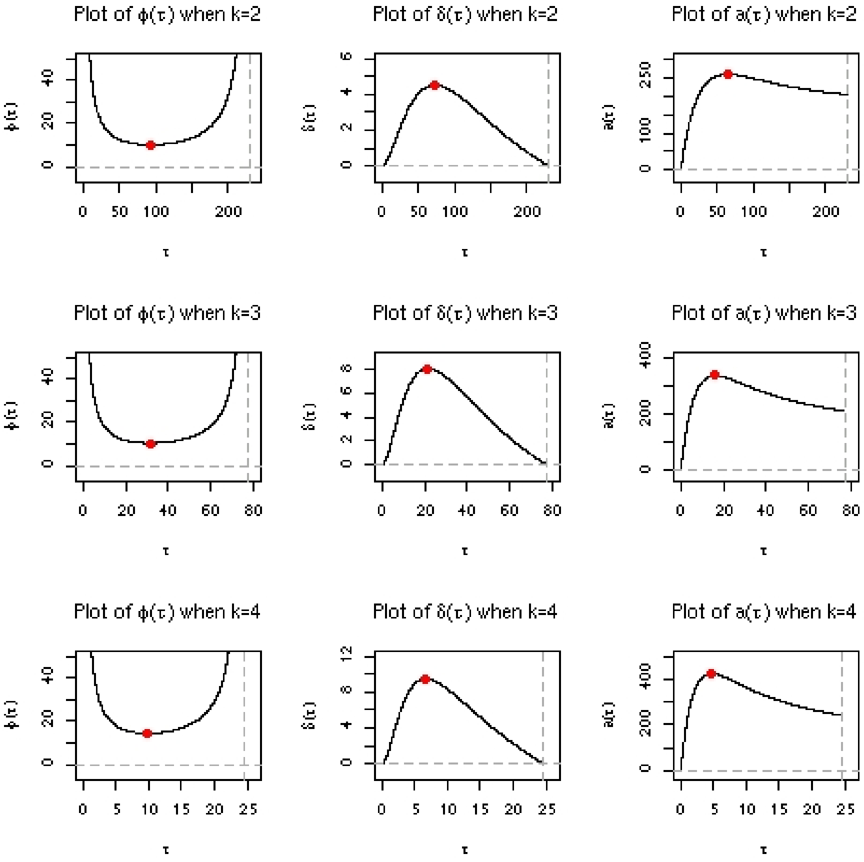
-

Table 3.1: Optimal Stress Change Points under the Large Sample Asymptotics with the Overall PC Proportion being 10%

		$k = 2$			$k = 3$			$k = 4$		
$\pi_i = 0.1$		τ_V^*	τ_D^*	τ_A^*	τ_V^*	τ_D^*	τ_A^*	τ_V^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	91.6	60.6	30.9	10.1	6.6	3.1	1.0	0.7	0.3
	$\rho = 0.3$	93.6	72.7	64.1	31.4	21.6	16.2	9.9	6.7	4.7
	$\rho = 0.5$	95.1	81.2	87.7	45.5	34.6	30.9	21.4	15.9	13.2
$\theta_1 = 300$	$\rho = 0.1$	274.9	181.7	92.8	30.4	19.9	9.2	2.9	2.1	1.0
	$\rho = 0.3$	280.7	218.0	192.4	94.2	64.7	48.7	29.6	20.0	14.1
	$\rho = 0.5$	285.4	243.5	263.0	136.6	103.8	92.8	64.1	47.7	39.5
$\theta_1 = 500$	$\rho = 0.1$	458.2	302.9	154.7	50.7	33.1	15.4	4.8	3.4	1.6
	$\rho = 0.3$	467.8	363.3	320.6	157.0	107.9	81.1	49.3	33.4	23.5
	$\rho = 0.5$	475.7	405.8	438.3	227.7	173.0	154.7	106.7	79.6	65.9

Table 3.2: Optimal Stress Change Points under the Large Sample Asymptotics with the Overall PC Proportion being 20%

$\pi_i = 0.2$		$k = 2$			$k = 3$			$k = 4$		
		τ_V^*	τ_D^*	τ_A^*	τ_V^*	τ_D^*	τ_A^*	τ_V^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	76.3	52.3	29.5	7.2	5.1	2.8	0.6	0.5	0.3
	$\rho = 0.3$	77.9	63.1	59.1	20.8	16.3	13.9	5.0	4.2	3.6
	$\rho = 0.5$	78.4	69.3	79.0	30.0	25.3	25.4	10.8	9.4	9.4
$\theta_1 = 300$	$\rho = 0.1$	228.8	156.9	88.4	21.5	15.4	8.5	1.7	1.4	0.8
	$\rho = 0.3$	233.6	189.2	177.3	62.5	49.0	41.6	15.0	12.5	10.8
	$\rho = 0.5$	235.3	207.9	237.0	90.1	76.0	76.1	32.4	28.2	28.1
$\theta_1 = 500$	$\rho = 0.1$	381.3	261.5	147.4	35.9	25.7	14.2	2.9	2.3	1.4
	$\rho = 0.3$	389.4	315.3	295.5	104.2	81.7	69.4	25.0	20.8	17.9
	$\rho = 0.5$	392.2	346.6	395.0	150.2	126.6	126.8	54.0	47.1	46.8



Plot of $\phi(\tau)$ when k=2

Plot of $\delta(\tau)$ when k=2

Plot of $a(\tau)$ when k=2

Plot of $\phi(\tau)$ when k=3

Plot of $\delta(\tau)$ when k=3

Plot of $a(\tau)$ when k=3

Plot of $\phi(\tau)$ when k=4

Plot of $\delta(\tau)$ when k=4

Plot of $a(\tau)$ when k=4

Figure 3.1: Plots of the Objective Functions for Each Optimality Criterion under the Large Sample Asymptotics with $\pi_i = 0.1$, $\theta_1 = 100$, and $\rho = 0.3$

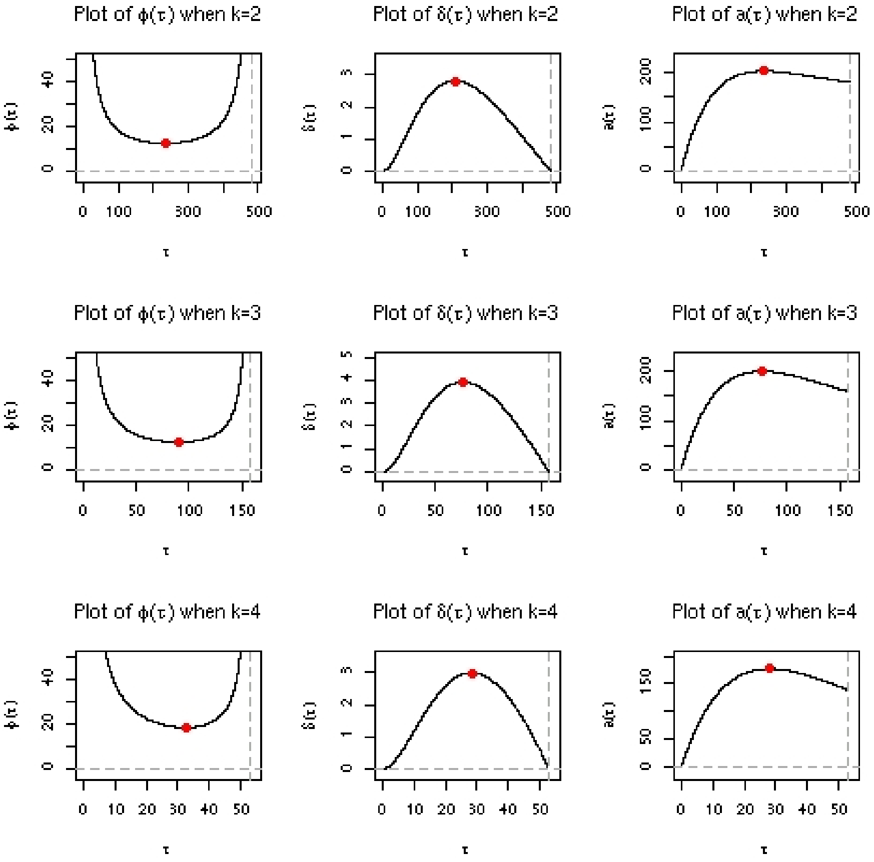


Figure 3.2: Plots of the Objective Functions for Each Optimality Criterion under the Large Sample Asymptotics with $\pi_i = 0.2$, $\theta_1 = 300$, and $\rho = 0.5$

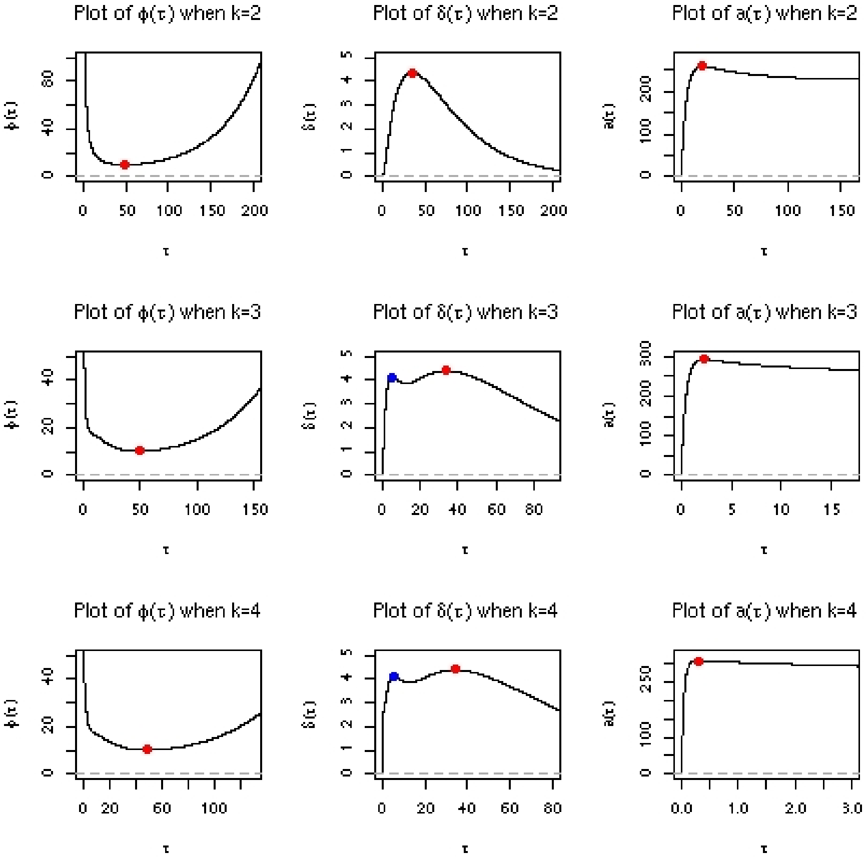


Figure 3.3: Plots of the Objective Functions for Each Optimality Criterion under the Modification of $c_i = (N_i - n_i)\pi_i^*$ with $\pi_i^* = 0.3$, $\theta_1 = 50$, and $\rho = 0.1$

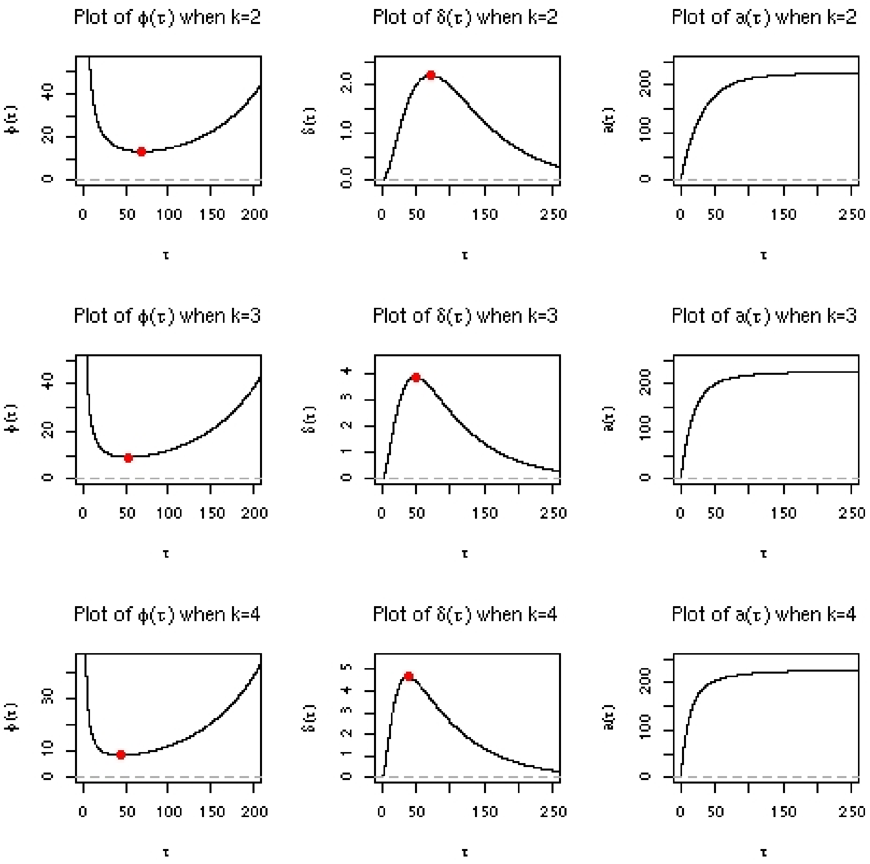


Figure 3.4: Plots of the Objective Functions for Each Optimality Criterion under the Modification of $c_i = (N_i - n_i)\pi_i^*$ with $\pi_i^* = 0.5$, $\theta_1 = 70$, and $\rho = 0.7$

Table 3.3: Fixed PC Proportions under the Modification of $c_i = (N_i - n_i)\pi_i^*$ for the Expected Overall PC Proportion at 10%

$\pi_i = 0.1$		$k = 2$			$k = 3$						$k = 4$								
Optimality		V	D	A	V		D		A		V			D			A		
		π_1^*	π_1^*	π_1^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*	π_3^*	π_1^*	π_2^*	π_3^*	π_1^*	π_2^*	π_3^*
$\theta_1 = 100$	$\rho = 0.1$	0.25	0.18	0.14	0.11	0.34	0.11	0.23	0.10	0.16	0.10	0.12	0.37	0.10	0.12	0.27	0.10	0.12	0.18
	$\rho = 0.3$	0.25	0.21	0.19	0.14	0.45	0.12	0.29	0.12	0.23	0.11	0.17	0.62	0.11	0.15	0.37	0.10	0.14	0.27
	$\rho = 0.5$	0.26	0.23	0.24	0.16	0.47	0.14	0.33	0.14	0.29	0.12	0.22	0.65	0.12	0.18	0.42	0.11	0.17	0.34
$\theta_1 = 300$	$\rho = 0.1$	0.25	0.18	0.14	0.11	0.34	0.11	0.23	0.10	0.16	0.10	0.12	0.37	0.10	0.12	0.27	0.10	0.12	0.18
	$\rho = 0.3$	0.25	0.21	0.19	0.14	0.45	0.12	0.29	0.12	0.23	0.11	0.17	0.62	0.11	0.15	0.37	0.10	0.14	0.27
	$\rho = 0.5$	0.26	0.23	0.24	0.16	0.47	0.14	0.33	0.14	0.29	0.12	0.22	0.65	0.12	0.18	0.42	0.11	0.17	0.34
$\theta_1 = 500$	$\rho = 0.1$	0.25	0.18	0.14	0.11	0.34	0.11	0.23	0.10	0.16	0.10	0.12	0.37	0.10	0.12	0.27	0.10	0.12	0.18
	$\rho = 0.3$	0.25	0.21	0.19	0.14	0.45	0.12	0.29	0.12	0.23	0.11	0.17	0.62	0.11	0.15	0.37	0.10	0.14	0.27
	$\rho = 0.5$	0.26	0.23	0.24	0.16	0.47	0.14	0.33	0.14	0.29	0.12	0.22	0.65	0.12	0.18	0.42	0.11	0.17	0.34

Table 3.4: Fixed PC Proportions under the Modification of $c_i = (N_i - n_i)\pi_i^*$ for the Expected Overall PC Proportion at 20%

$\pi_i = 0.2$		$k = 2$			$k = 3$						$k = 4$								
Optimality		V	D	A	V		D		A		V			D			A		
		π_1^*	π_1^*	π_1^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*	π_3^*	π_1^*	π_2^*	π_3^*	π_1^*	π_2^*	π_3^*
$\theta_1 = 100$	$\rho = 0.1$	0.43	0.34	0.27	0.21	0.56	0.21	0.45	0.21	0.34	0.20	0.27	0.65	0.20	0.26	0.56	0.20	0.26	0.46
	$\rho = 0.3$	0.44	0.38	0.36	0.25	0.65	0.24	0.53	0.23	0.47	0.21	0.31	0.80	0.21	0.30	0.69	0.21	0.29	0.62
	$\rho = 0.5$	0.44	0.40	0.44	0.27	0.67	0.26	0.58	0.26	0.58	0.22	0.36	0.85	0.22	0.34	0.75	0.22	0.34	0.75
$\theta_1 = 300$	$\rho = 0.1$	0.43	0.34	0.27	0.21	0.56	0.21	0.45	0.21	0.34	0.20	0.27	0.65	0.20	0.26	0.56	0.20	0.26	0.46
	$\rho = 0.3$	0.44	0.38	0.36	0.25	0.65	0.24	0.53	0.23	0.47	0.21	0.31	0.80	0.21	0.30	0.69	0.21	0.29	0.62
	$\rho = 0.5$	0.44	0.40	0.44	0.27	0.67	0.26	0.58	0.26	0.58	0.22	0.36	0.85	0.22	0.34	0.75	0.22	0.34	0.75
$\theta_1 = 500$	$\rho = 0.1$	0.43	0.34	0.27	0.21	0.56	0.21	0.45	0.21	0.34	0.20	0.27	0.65	0.20	0.26	0.56	0.20	0.26	0.46
	$\rho = 0.3$	0.44	0.38	0.36	0.25	0.65	0.24	0.53	0.23	0.47	0.21	0.31	0.80	0.21	0.30	0.69	0.21	0.29	0.62
	$\rho = 0.5$	0.44	0.40	0.44	0.27	0.67	0.26	0.58	0.26	0.58	0.22	0.36	0.85	0.22	0.34	0.75	0.22	0.34	0.75

Table 3.5: Optimal Stress Change Points for the Simple Step-Stress Testing ($k = 2$) under the Condition of $N_k > 0$ with the Expected Overall PC Proportion being 10%

$\pi_1 = 0.1$		$n = 5$			$n = 10$			$n \geq 20$		
		τ_V^*	τ_D^*	τ_A^*	τ_V^*	τ_D^*	τ_A^*	τ_V^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	119.6	71.2	DNE (31.4)	93.6	60.8	DNE (30.9)	91.6	60.6	30.9
	$\rho = 0.3$	123.2	90.6	DNE	95.7	73.3	DNE (64.6)	93.6	72.7	64.1
	$\rho = 0.5$	130.5	113.6	DNE	97.7	82.5	DNE (92.8)	95.1	81.2	87.7
$\theta_1 = 300$	$\rho = 0.1$	358.7	213.7	DNE (94.2)	280.7	182.5	DNE (92.8)	274.9	181.7	92.8
	$\rho = 0.3$	369.7	271.7	DNE	287.2	220.0	DNE (193.7)	280.7	218.0	192.4
	$\rho = 0.5$	391.6	340.9	DNE	293.1	247.6	DNE (278.4)	285.4	243.5	263.0
$\theta_1 = 500$	$\rho = 0.1$	597.9	356.2	DNE (157.0)	467.9	304.1	DNE (154.7)	458.2	302.9	154.7
	$\rho = 0.3$	616.1	452.9	DNE	478.7	366.7	DNE (322.9)	467.8	363.3	320.6
	$\rho = 0.5$	652.6	568.1	DNE	488.5	412.7	DNE (463.9)	475.7	405.8	438.3

Table 3.6: Optimal Stress Change Points for the Simple Step-Stress Testing ($k = 2$) under the Condition of $N_k > 0$ with the Expected Overall PC Proportion being 20%

$\pi_1 = 0.2$		$n = 5$			$n = 10$			$n \geq 20$		
		τ_V^*	τ_D^*	τ_A^*	τ_V^*	τ_D^*	τ_A^*	τ_V^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	87.1	56.9	DNE (29.9)	76.8	52.4	29.5	76.3	52.3	29.5
	$\rho = 0.3$	89.8	71.2	DNE	78.5	63.3	59.4	77.9	63.1	59.1
	$\rho = 0.5$	91.9	81.8	DNE	79.1	69.8	81.3	78.4	69.3	79.0
$\theta_1 = 300$	$\rho = 0.1$	261.3	170.8	DNE (89.6)	230.4	157.1	88.4	228.8	156.9	88.4
	$\rho = 0.3$	269.3	213.6	DNE	235.5	189.9	178.1	233.6	189.2	177.3
	$\rho = 0.5$	275.8	245.3	DNE	237.4	209.3	243.8	235.3	207.9	237.0
$\theta_1 = 500$	$\rho = 0.1$	435.5	284.6	DNE (149.3)	384.0	261.9	147.4	381.3	261.5	147.4
	$\rho = 0.3$	448.8	356.0	DNE	392.5	316.5	296.8	389.4	315.3	295.5
	$\rho = 0.5$	459.6	408.8	DNE	395.7	348.8	406.3	392.2	346.6	395.0

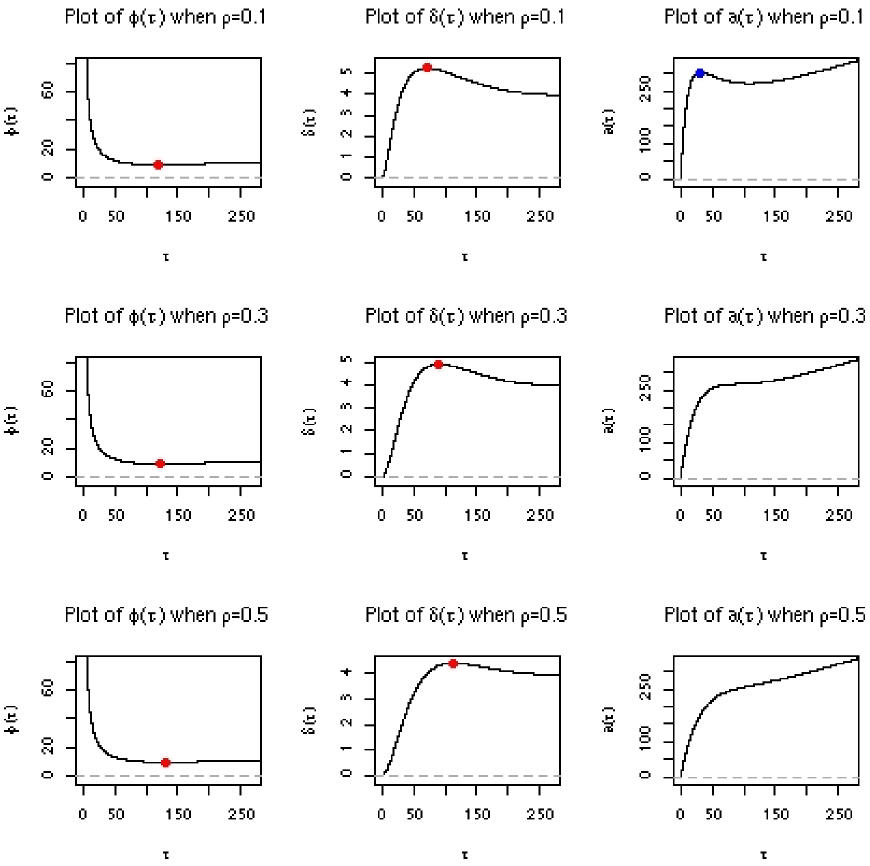


Figure 3.5: Plots of the Objective Functions for Each Optimality Criterion for the Simple Step-Stress Testing ($k = 2$) under the Condition of $N_k > 0$ with $n = 5$, $\theta_1 = 100$, and the Expected Overall PC Proportion at 10%

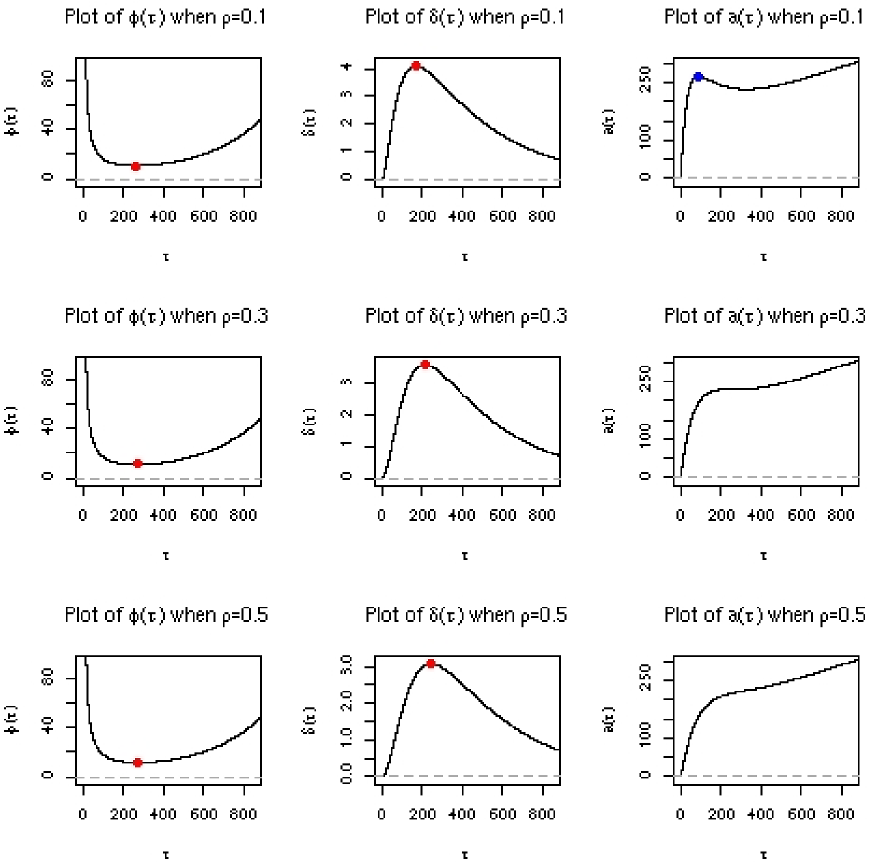


Figure 3.6: Plots of the Objective Functions for Each Optimality Criterion for the Simple Step-Stress Testing ($k = 2$) under the Condition of $N_k > 0$ with $n = 5$, $\theta_1 = 300$, and the Expected Overall PC Proportion at 20%

Table 3.7: Fixed PC Proportions π_1^* for the Simple Step-Stress Testing ($k = 2$) under the Condition of $N_k > 0$ with the Expected Overall PC Proportion being 10%

$\pi_1 = 0.1$		$n = 5$			$n = 10$			$n \geq 20$		
Optimality		V	D	A	V	D	A	V	D	A
$\theta_1 = 100$	$\rho = 0.1$	0.28	0.20	DNE (0.14)	0.25	0.18	DNE (0.14)	0.25	0.18	0.14
	$\rho = 0.3$	0.28	0.23	DNE	0.26	0.21	DNE (0.19)	0.25	0.21	0.19
	$\rho = 0.5$	0.29	0.27	DNE	0.26	0.23	DNE (0.25)	0.26	0.23	0.24
$\theta_1 = 300$	$\rho = 0.1$	0.28	0.20	DNE (0.14)	0.25	0.18	DNE (0.14)	0.25	0.18	0.14
	$\rho = 0.3$	0.28	0.23	DNE	0.26	0.21	DNE (0.19)	0.25	0.21	0.19
	$\rho = 0.5$	0.29	0.27	DNE	0.26	0.23	DNE (0.25)	0.26	0.23	0.24
$\theta_1 = 500$	$\rho = 0.1$	0.28	0.20	DNE (0.14)	0.25	0.18	DNE (0.14)	0.25	0.18	0.14
	$\rho = 0.3$	0.28	0.23	DNE	0.26	0.21	DNE (0.19)	0.25	0.21	0.19
	$\rho = 0.5$	0.29	0.27	DNE	0.26	0.23	DNE (0.25)	0.26	0.23	0.24

Table 3.8: Fixed PC Proportions π_1^* for the Simple Step-Stress Testing ($k = 2$) under the Condition of $N_k > 0$ with the Expected Overall PC Proportion being 20%

$\pi_1 = 0.2$		$n = 5$			$n = 10$			$n \geq 20$		
Optimality		V	D	A	V	D	A	V	D	A
$\theta_1 = 100$	$\rho = 0.1$	0.45	0.35	DNE (0.27)	0.43	0.34	0.27	0.43	0.34	0.27
	$\rho = 0.3$	0.46	0.39	DNE	0.44	0.38	0.36	0.44	0.38	0.36
	$\rho = 0.5$	0.46	0.43	DNE	0.44	0.40	0.45	0.44	0.40	0.44
$\theta_1 = 300$	$\rho = 0.1$	0.45	0.35	DNE (0.27)	0.43	0.34	0.27	0.43	0.34	0.27
	$\rho = 0.3$	0.45	0.39	DNE	0.44	0.38	0.36	0.44	0.38	0.36
	$\rho = 0.5$	0.46	0.43	DNE	0.44	0.40	0.45	0.44	0.40	0.44
$\theta_1 = 500$	$\rho = 0.1$	0.45	0.35	DNE (0.27)	0.43	0.34	0.27	0.43	0.34	0.27
	$\rho = 0.3$	0.45	0.39	DNE	0.44	0.38	0.36	0.44	0.38	0.36
	$\rho = 0.5$	0.46	0.43	DNE	0.44	0.40	0.45	0.44	0.40	0.44

4 REFERENCES

1. **R. Miller and W. Nelson, "Optimum simple step-stress plans for accelerated life testing,"** *IEEE Transactions on Reliability*, **vol. 32, pp. 59-65, 1983.**

2. **W. Nelson, "Accelerated life testing - step-stress models and data analysis,"** *IEEE Transactions on Reliability*, **vol. R-29, pp. 103-108, 1980.**

3. **D. S. Bai, M. S. Kim, and S. H. Lee, "Optimum simple step-stress accelerated life tests with censoring,"** *IEEE Transactions on Reliability*, **vol. 38, pp. 528-532, 1989.**

4. **I. H. Khamis and J. J. Higgins, "Optimum 3-step step-stress tests,"** *IEEE Transactions on Reliability*, **vol. 45, pp. 341-345, 1996.**

5. **I. H. Khamis and J. J. Higgins, "A new model for step-stress testing,"** *IEEE Transactions on Reliability*, **vol. 47, pp. 131-134, 1998.**

6. **I. H. Khamis, "Optimum M-step, step-stress test with k stress variables,"** *Communications in Statistics - Computation and Simulation*, **vol. 26, pp. 1301-1313, 1997.**

7. **K. P. Yeo and L. C. Tang, "Planning step-stress life-test with a target acceleration-factor,"** *IEEE Transactions on Reliability*, **vol. 48, pp. 61-67, 1999.**

8. C. Xiong, “Inferences on a simple step-stress model with Type-II censored exponential data,” *IEEE Transactions on Reliability*, vol. 47, pp. 142-146, 1998.

9. C. Xiong and G. A. Milliken, “Step-stress life-testing with random stress-change times for exponential data,” *IEEE Transactions on Reliability*, vol. 48, pp. 141-148, 1999.

10. N. Balakrishnan, D. Kundu, H. K. T. Ng, and N. Kannan, “Likelihood inference for a simple step-stress model with Type-II censoring,” *Journal of Quality Technology*, (to appear).

11. E. Gouno, A. Sen, and N. Balakrishnan, “Optimal step-stress test under progressive Type-I censoring,” *IEEE Transactions on Reliability*, vol. 53, pp. 383-393, 2004.

12. E. Gouno and N. Balakrishnan, “Step-stress accelerated life test,” in *Handbook of Statistics, vol. 20: Advances in Reliability*, N. Balakrishnan and C. R. Rao, Eds. Amsterdam: North-Holland, 2001, pp. 623-639.

13. A. C. Cohen, *Truncated and Censored Samples: Theory and Applications*. New York: Marcel Dekker, 1991.